

Energy dissipation statistics in a shell model of turbulence

G. Boffetta,¹ A. Celani,^{1,2} and D. Roagna¹

¹*Dipartimento di Fisica Generale, Università di Torino and INFN Unità di Torino Università, Via Pietro Giuria 1, 10125 Torino, Italy*

²*Observatoire de la Côte d'Azur, BP 4229, F-06304 Nice Cedex 04, France*

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The Reynolds number dependence of the statistics of energy dissipation is investigated in a shell model of fully developed turbulence. The results are in agreement with a model which accounts for fluctuations of the dissipative scale with the intensity of energy dissipation. It is shown that the assumption of a fixed dissipative scale leads to a different scaling with Reynolds which is not compatible with numerical results.

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One of the most important problems in fully developed turbulence is the description of the energy transfer mechanism. In stationary situations, the energy injected at the large scale ℓ_0 , transfers at rate $\bar{\epsilon}$ down to the dissipative scale η , where it is removed at the same rate by viscous dissipation. The fundamental assumption in the study of fully developed turbulence is that in the limit of very high Reynolds numbers Re , the energy dissipation $\bar{\epsilon}$ becomes independent of Re (i.e., of the viscosity, being $Re = u_0 \ell_0 / \nu$, with u_0 a typical large scale velocity) [1–4]. In the same limit, the Kolmogorov theory predicts universal scaling of the velocity structure functions in the inertial range of scales $\eta < \ell < \ell_0$:

$$S_q(\ell) \equiv \langle [\delta u(\ell)]^q \rangle \sim u_0^q \left(\frac{\ell}{\ell_0} \right)^{\zeta_q} \quad (1)$$

with exponents $\zeta_q = q/3$.

Several decades of experimental and numerical investigation have shown that scaling laws (1) are indeed observed but with exponents ζ_q corrected with respect to the Kolmogorov prediction [5]. This is the essence of the intermittency problem, which has received a lot of attention in the modern approach of the study of fully developed turbulence.

Experiments have shown that intermittency also affects energy dissipation statistics [6] which is not uniform in the turbulent domain. A phenomenological description of intermittency is the multifractal model [7]. This model introduces a continuous set of scaling exponents h which relate the velocity fluctuations entering in Eq. (1) with the large scale velocity u_0 :

$$\delta u(\ell) \sim u_0 \left(\frac{\ell}{\ell_0} \right)^h \quad (2)$$

The exponent h is realized with a probability $(\ell/\ell_0)^{Z(h)}$ where $Z(h)$ is the codimension of the fractal set on which the h -scaling holds. The scaling exponents of structure functions (1) are obtained by a steepest descent argument over exponents h :

$$\zeta_q = \inf_h [qh + Z(h)]. \quad (3)$$

The scaling region is bounded from below by the Kolmogorov dissipation scale η at which dissipation starts to dominate, i.e., the local Reynolds number is of order 1:

$$\frac{\eta \delta u(\eta)}{\nu} \sim 1. \quad (4)$$

At variance with the Kolmogorov theory, in the multifractal description of intermittent turbulence, the dissipative scale is a fluctuating quantity. This implies a series of consequences which have been investigated in past years [8,9]. As we shall see later the description of the fluctuations of the dissipative scale is crucial for the correct evaluation of the Reynolds number dependence.

In this paper we are interested in the dependence of the statistics of energy dissipation on the Reynolds number. The physical picture is that dissipation becomes more and more intermittent as the Reynolds number increases. Assuming that the multifractal description can be pushed down to the dissipative scale, one predicts for the moment of energy dissipation a power-law dependence on Re , with exponents related to the structure function exponents (1) [10,2]. We will see that this prediction is rather natural and confirmed by numerical simulations on a shell model.

The dimensional argument for the prediction goes as follows. In a dimensional approach, the energy dissipation is evaluated as

$$\epsilon = \nu \sum_{\alpha, \beta} \left(\frac{\partial u_\alpha}{\partial x_\beta} \right)^2 \sim \nu \left(\frac{\delta u(\eta)}{\eta} \right)^2. \quad (5)$$

From Eqs. (2) and (4) one has that $\eta \sim \ell_0 Re^{-(1/1+h)}$. Inserting in Eq. (5) and computing the average of the different moments, one ends with the expression

$$\langle \epsilon^p \rangle \sim \bar{\epsilon}^p \int d\mu(h) Re^{-[3ph - p + Z(h)]/(1+h)} \sim \bar{\epsilon}^p Re^{-\theta_p}, \quad (6)$$

where the integral has been evaluated by a steepest descent argument (assuming $Re \rightarrow \infty$) and

$$\theta_p = \inf_h \left[\frac{3ph - p + Z(h)}{1+h} \right]. \quad (7)$$

The standard inequality in the multifractal model (following from the exact result $\zeta_3 = 1$), $Z(h) \geq 1 - 3h$, implies for Eq. (7) $\theta(1) = 0$ which is nothing but the request of finite

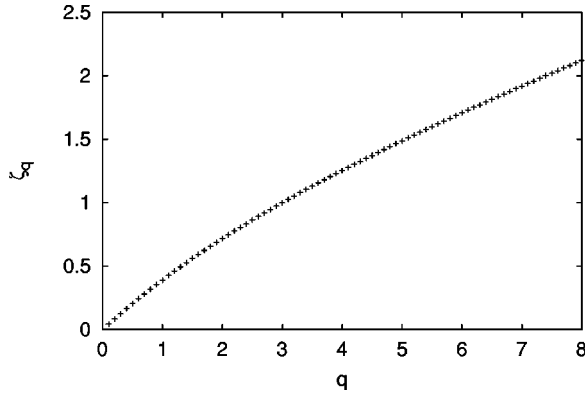


FIG. 1. Structure function exponents ζ_q obtained from simulation of the shell model. The number of shells is $N=24$ and $\nu = 10^{-7}$ corresponding to $\text{Re}=10^8$.

nonvanishing dissipation in the limit $\text{Re} \rightarrow \infty$. For $p > 1$, $\theta_p < 0$, i.e., the tail of the distribution of ϵ becomes wider with the Reynolds number.

Let us stress that the above argument is only a reasonable dimensional argument. It is essentially based on two assumptions: a physical one concerning the fluctuations of the dissipative scale according to Eq. (4), and a more formal one on the possibility of extending the multifractal description down to the dissipative scales. The two assumptions are independent: indeed, as we will see, it is possible to give different predictions by changing assumption (4) [11].

It would thus be important to address the problem with experiments or direct numerical simulation at high Reynolds numbers. Recent high resolution direct numerical simulation gives support to Eq. (7) [12], but the Reynolds number is not large enough to discriminate clearly between different predictions.

Shell models are extremely simplified models of turbulence. Nevertheless, they are deterministic, nonlinear dynamical models which display intermittency and anomalous scaling exponents reminiscent of real turbulence [13]. Their main advantage is that with shell models one can perform very accurate simulations at very high Reynolds numbers; for this reason they are thus natural candidates for a numerical investigation of Reynolds number dependence.

In shell models, the velocity fluctuations are represented by a single complex variable u_n on shells of geometrically

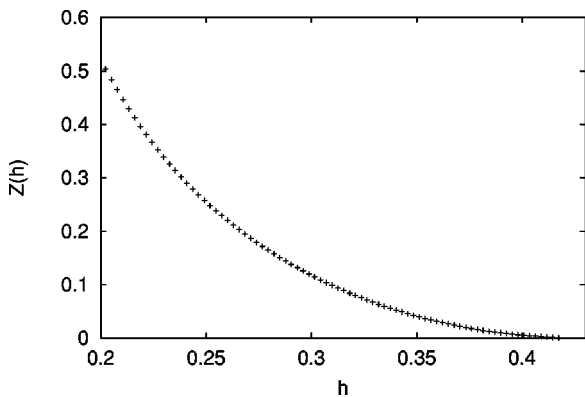


FIG. 2. $Z(h)$ computed by inverting the Legendre transformation from the data of Fig. 1.

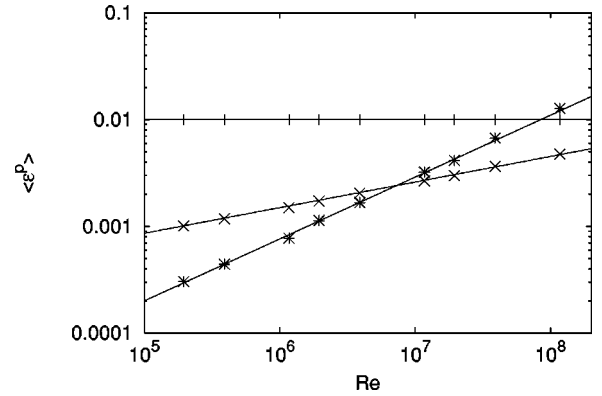


FIG. 3. Moments of energy dissipation $\langle \epsilon^p \rangle$ as function of Re for $p=1(+)$, $p=2(\times)$, and $p=3(*)$.

spaced wave number $k_n = k_0 2^n$. The particular model we adopt for our investigation is a recently introduced model which displays strong intermittency corrections [14]. The model equations are

$$\frac{du_n}{dt} = ik_n \left(u_{n+2} u_{n+1}^* - \frac{1}{4} u_{n+1} u_{n-1}^* + \frac{1}{8} u_{n-1} u_{n-2} \right) - \nu k_n^2 u_n + f_n, \quad (8)$$

where ν is the viscosity and f_n is a forcing term restricted to the first two shells. For $\nu = f_n = 0$ the model conserves the total energy $E = \sum_n |u_n|^2$. For simplicity, the forcing adopted for the present simulations is $f_n \propto 1/u_n^*$, which guarantees a constant energy input $\bar{\epsilon}$. The large scale Reynolds number of the simulation is estimated as $\text{Re} = \bar{\epsilon}^{1/3} / (\nu k_0^{4/3})$ and is numerically controlled by the value of the viscosity.

The chaotic dynamics is responsible for intermittency corrections to the structure functions exponent ζ_q , here defined by means of

$$S_q(n) = \langle |u_n|^q \rangle \sim k_n^{-\zeta_q}, \quad (9)$$

which are close to the experimental values [13]. In Fig. 1 we plot the spectrum of structure function exponents obtained from very long simulations. The multifractal codimension $Z(h)$ is numerically obtained from $\zeta(p)$ by inverting the Legendre transform (3). The result is shown in Fig. 2. We observe that, because of the strong intermittency in the model, it is numerically difficult to obtain statistical convergence of structure functions (9) of order $q > 8$. As a consequence, the minimum exponent for $Z(h)$ is $h_{\min} \approx 0.2$.

From the energy balance equation we have the instantaneous energy dissipation

$$\epsilon = 2\nu \sum_n k_n^2 |u_n|^2, \quad (10)$$

whose average is $\langle \epsilon \rangle = \bar{\epsilon}$ in stationary conditions.

We have performed very long simulations at different Reynolds numbers, starting from $\text{Re} = 2 \times 10^5$ up to $\text{Re} = 10^8$. For each simulation we computed the different moments of energy dissipation, $\langle \epsilon^p \rangle$. Shell models dynamics is characterized by strong bursts of energy dissipation which limits the possibility of computing with confidence high or-

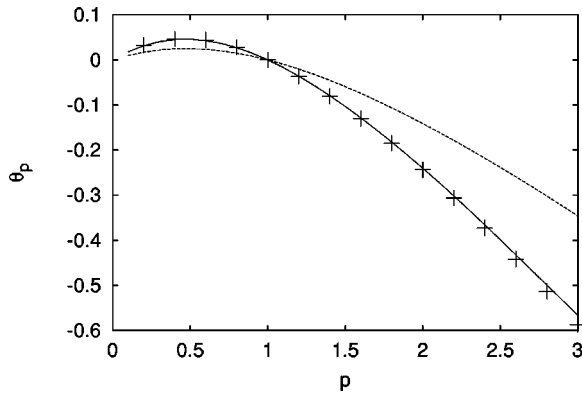


FIG. 4. Energy dissipation scaling exponents θ_p . Symbols represent the exponents obtained from the fit of Fig. 3. The continuous line is prediction (7) taking into account the fluctuations of the dissipative scale. The dashed line is the prediction obtained by assuming an average dissipative scale.

der moments. Here we are limited to moments $p \leq 8$. In Fig. 3 we plot the behavior of $\langle \epsilon^p \rangle$ as a function of Re for different values of p . The power-law behavior is evident and the scaling exponent θ_p can be estimated with good accuracy. By construction $\langle \epsilon \rangle = \bar{\epsilon}$ is independent of Re .

The scaling exponent θ_p (6) are plotted in Fig. 4, together with the multifractal prediction (7). Let us observe that, because the largest q in Eq. (9) is $q = 8$, the estimate of p in Eq.

(7) is limited to values less than $p \approx 2.5$. For higher p , the numerical evaluation of Eq. (7) feels the effect of the cutoff of h on h_{min} . Nevertheless, we have a rather large range of moments ($0 \leq p \leq 2.5$) over which numerical data display a perfect agreement with Eq. (7).

As discussed above, prediction (8) makes use of the fluctuating dissipative scale η . If, on the contrary, one assumes that the dissipation scale enters in Eq. (5) as an averaged quantity the prediction for θ_p is different: assuming $\nu \langle \delta u(\tilde{\eta})^2 \rangle / \tilde{\eta}^2 \sim \bar{\epsilon}$ as the definition of the (nonfluctuating) dissipation scale $\tilde{\eta}$ (this is the only choice that ensures that $\langle \epsilon \rangle \sim \text{Re}^0$), one ends up with $\tilde{\theta}_p = [p\zeta_2 - \zeta_{2p}] / [2 - \zeta_2]$ [11].

Our results allow us to discriminate between prediction (7) and the one obtained with a nonfluctuating dissipative scale. Figure 4 shows that the numerical θ_p is definitely not compatible with the latter alternative, whereas it supports with good accuracy prediction (7).

In conclusion, it is an expected consequence of the existence of intermittency in the energy transfer that the dissipative scale η fluctuates according to the local intensity of energy dissipation, being smaller where the dissipation is stronger and vice versa. The fluctuations of the inner scale of turbulence reflect onto the Reynolds dependence of the statistics of energy dissipation. Long numerical simulations of shell models confirm with great accuracy the validity of the multifractal model, which accounts for the fluctuations of η , and rule out alternative models which do not describe properly the correlations between η and ϵ .

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